Statistical Inference
Midterm Examination II

2017/11/27

Time: 1:20 pm – 5:00 pm

1. (10%) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ symmetric matrices. Show that

$$\lambda_{\min}(\boldsymbol{A}) \geq \lambda_{\min}(\boldsymbol{B}) - \|\boldsymbol{A} - \boldsymbol{B}\|,$$

where $\|\cdot\|$ denotes the spectral norm.

2. (20%) Consider the location-dispersion model

$$y_t = \boldsymbol{x}_t'\boldsymbol{\beta}_0 + \sigma_t \epsilon_t,$$

where $\sigma_t^2 = \exp\{\boldsymbol{x}_t'\boldsymbol{\alpha}_0\}$ and $\epsilon_t \overset{\text{i.i.d.}}{\sim} (0,1)$. Define

$$\ell(\boldsymbol{\eta}) = \ell(\boldsymbol{\beta}, \boldsymbol{\alpha}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n} \boldsymbol{x}_t'\boldsymbol{\alpha} - \frac{1}{2}\sum_{t=1}^{n}(y_t - \boldsymbol{x}_t'\boldsymbol{\beta})^2 e^{-\boldsymbol{x}_t'\boldsymbol{\alpha}},$$

and

$$\hat{\boldsymbol{\eta}} = (\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}}) = \underset{\boldsymbol{\eta} \in \boldsymbol{\Theta}}{\operatorname{argmax}}\, \ell(\boldsymbol{\eta}),$$

where $\boldsymbol{\Theta}$ is the parameter space. Please construct an asymptotic 95% confidence interval for $\boldsymbol{v}'\boldsymbol{\eta}$, where $\boldsymbol{v}$ is a known vector.

3. (20%) Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$ be a random matrix in which $\boldsymbol{X}$, $\boldsymbol{X}_1$, and $\boldsymbol{X}_2$ are $n \times p$, $n \times k$, and $n \times (p-k)$ matrices, respectively. Assume

$$\frac{1}{n}\boldsymbol{X}'\boldsymbol{X} \xrightarrow{pr.} \boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22} \end{pmatrix},$$

where $\boldsymbol{R}$ is a positive definite non-random matrix, and $\boldsymbol{R}_{11}$ and $\boldsymbol{R}_{22}$ are $k \times k$ and $(p-k) \times (p-k)$ matrices, respectively. Assume also that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} \boldsymbol{x}_t \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{R}),$$

where $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) = \boldsymbol{X}'$. Find the limiting distribution of

$$\frac{1}{\sqrt{n}}\boldsymbol{X}_1'(\boldsymbol{I} - \boldsymbol{M}_2)\boldsymbol{1},$$

where $\boldsymbol{M}_2$ is the orthogonal projection matrix onto $C(\boldsymbol{X}_2)$, the column space of $\boldsymbol{X}_2$, and $\boldsymbol{1}$ denotes the n-dimensional vector of 1's.

4. (10%) Assume $\hat{\sigma}_1^2 \xrightarrow{pr.} \sigma^2$, $\hat{\sigma}_2^2 \xrightarrow{pr.} \sigma^2$, and

$$n(\hat{\sigma}_1^2 - \hat{\sigma}_2^2) = O_p(1).$$

Show that

$$n(\log \hat{\sigma}_1^2 - \log \hat{\sigma}_2^2) = O_p(1).$$

5. (10%) Assume $\{X_1, \ldots, X_n\}$ is a random sample generated according to the probability density function $f(x)$, where $f(x)$ satisfies $f(x) = 0$ for all $x \le 0$, and

$$\lim_{x \downarrow 0} \frac{f(x)}{bx^{\alpha-1}} = 1 \text{ for some } b > 0 \text{ and } \alpha > 0.$$

Find the limiting distribution of

$$n^{\frac{1}{\alpha}} X_{(1)},$$

where $X_{(1)}$ is the smallest order statistic of $\{X_1, \ldots, X_n\}$.

6. (10%) Define

$$\hat{\boldsymbol{\eta}}^{\text{New}} = \tilde{\boldsymbol{\eta}} - \left(\frac{\partial^2}{\partial \eta_i \partial \eta_j} \ell(\tilde{\boldsymbol{\eta}}) \right)^{-1} \left(\frac{\partial}{\partial \boldsymbol{\eta}} \ell(\tilde{\boldsymbol{\eta}}) \right),$$

where $\ell(\boldsymbol{\eta})$ is the log-likelihood function and $\tilde{\boldsymbol{\eta}}$ is an initial estimate of the true parameter $\boldsymbol{\eta}_0$. Show that if

$$\|\tilde{\boldsymbol{\eta}} - \boldsymbol{\eta}_0\| = O_p(n^{-\frac{1}{q}}) \text{ for some } q \ge 4,$$

then

$$\|\hat{\boldsymbol{\eta}}^{\text{New}} - \boldsymbol{\eta}_0\| = O_p(n^{-\frac{2}{q}}).$$

(You can choose sufficient conditions that you prefer.)

7. (20%) Define

$$y_t(\theta) = \sum_{j=0}^{K} e^{-j\theta} X_{t-j},$$

where $X_t \overset{\text{i.i.d.}}{\sim} N(0,1)$, $K \ge 1$ is a fixed integer, and $\theta > 0$. Show that

$$E\left(\sup_{\theta \in [a,b]} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} y_t(\theta) \right)^2 \right) < C < \infty,$$

where $0 < a < b < \infty$ and C is some positive constant, which implies

$$\sup_{\theta \in [a,b]} \frac{1}{n} \sum_{t=1}^{n} y_t(\theta) = o_p(1).$$